

RELATIVE AGING OF DISTRIBUTIONS

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We consider the reliability of one random variable relative to another, when the variables are continuous and take values in an interval $[a, b)$. We give definitions and characterizations for the exponential property and the standard aging properties IFR, IFRA, and NBU. The exponential property defines an equivalence relation on the distributions, and then each of these aging properties defines a partial order on the distributions, modulo the exponential equivalence. We give a set of conditions that must be satisfied for a general aging property to define such a partial order and show that these conditions are not satisfied by the NBUE property. Several parametric families of distributions are considered.

1. INTRODUCTION

The purpose of this paper is to study the reliability of one random variable relative to another, when the variables are continuous and take values in an interval $[a, b)$. It turns out that this is equivalent to studying the reliability of a distribution relative to a semigroup operation on $[a, b)$ that is isomorphic to $+$ on $[0, \infty)$ and corresponds to a transformation of time. Ultimately, the definitions reduce to the standard ones (see [1,3,4]), but still this point of view provides some fresh insights. For example, the exponential property defines an equivalence relation on the class of distributions, and then the IFR, IFRA, and NBU aging properties each defines a partial order on

the distributions, modulo the exponential equivalence. This suggests that any "strong" aging property should define a partial order, when extended to relative aging. We give a set of conditions that must be satisfied for this to be true and then show that these conditions are not satisfied by the NBUE property. Several parametric families of distributions are considered in detail, including the Weibull family and the modified extreme value family.

2. THE SEMIGROUP OF A DISTRIBUTION

Fix an interval $[a, b]$ with $-\infty < a < b \leq \infty$. Suppose that X is a random variable with a continuous distribution whose support is the closure of $[a, b]$ (i.e., either $[a, b]$ if $b < \infty$ or $[a, \infty)$ if $b = \infty$). For brevity, we will refer to this as a continuous distribution on $[a, b]$. The distribution of X must satisfy

$$P(X > x) = e^{-R(x)}, \quad x \in [a, b], \quad (1)$$

where R is a strictly increasing, continuous function from $[a, b]$ onto $[0, \infty)$. Of course, the function in (1) is the reliability function of X , and R is the cumulative failure rate function of X (in the usual sense).

We define the operation $*$ on $[a, b]$ associated with R as follows:

$$x * y = R^{-1}[R(x) + R(y)] \quad \text{for } x, y \in [a, b]. \quad (2)$$

With this definition, $([a, b], *)$ is a semigroup isomorphic to $([0, \infty), +)$ and R is an isomorphism. It is a special case of a *temporal semigroup*, as studied in [7] and [5], and amounts to a transformation of time according to R . Of course, all of the standard algebraic operations on $[0, \infty)$ have their isomorphic counterparts on $[a, b]$. For example,

$$x^{*\alpha} = R^{-1}[\alpha R(x)] \quad \text{for } x \in [a, b], \quad \alpha \geq 0.$$

Moreover, two functions R and S generate the same semigroup operation if and only if $S = cR$ for some constant $c > 0$.

The *order* on $[a, b]$ induced by the semigroup operation $*$ (in the sense of [7]) is the ordinary order \leq . By definition, this means that

$$x \leq y \text{ if and only if } x * z = y \quad \text{for some } z \in [a, b].$$

Finally, we define the measure r on $[a, b]$ associated with R by

$$r(A) = l[R(A)] \quad \text{for measurable } A \subseteq [a, b], \quad (3)$$

where l is Lebesgue measure. This is an *invariant* (or *Harr measure*) for $*$ in the sense that

$$r(x * A) = r(A) \quad \text{for } x \in [a, b] \text{ and measurable } A \subseteq [a, b].$$

3. RELIABILITY DEFINITIONS

Suppose now that X and Y are continuous random variables on $[a, b]$. Let R and S , respectively, denote the cumulative failure rate functions of X and Y , as in Eq. (1); let $*$ and \circ , respectively, denote the semigroup operations associated with R and S , as in

Eq. (2); and let r and s , respectively, denote the corresponding invariant measures, as in Eq. (3).

To study the reliability of X relative to Y , it is natural to study the reliability of X relative to

$$([a, b], \circ, \leq, s)$$

as a model of time.

First, the *reliability function* of X is the function given in (1). It is independent of Y because it depends only on the order on $[a, b]$, which is the same for all of the semigroup operations.

If R and S have positive derivatives on (a, b) (R and S are nonsingular), then X has a density function relative to the measure s ; it is given by

$$f(x) = e^{-R(x)} \frac{R'(x)}{S'(x)} \quad \text{for } x \in (a, b) \text{ with } S'(x) > 0.$$

By analogy with the standard case, the failure rate of X is the density function divided by the reliability function. Thus, we define R'/S' to be the *failure rate* of X relative to Y . Next, the cumulative failure rate at $x \in [a, b]$ is obtained by integrating the failure rate by the invariant measure s over $[a, x]$; this simply gives $R(x)$. Thus, regardless of whether R or S is nonsingular, we define R to be the *cumulative failure rate* of X . As with the reliability function, the cumulative failure rate function is independent of Y because it depends only on the order on $[a, b]$.

Finally, the failure rate average over $[a, x]$ is the cumulative failure rate over $[a, x]$ divided by the length of this interval, as measured by the invariant measure s . This length is simply $S(x)$ and, hence, we define R/S to be the *failure rate average* of X relative to Y .

4. THE EXPONENTIAL PROPERTY

By analogy with the standard setting, we say that X is *exponential* relative to Y if the distribution of $x^{\circ(-1)} \circ X$ given $X \geq x$ is the same as the distribution of X (see [7]).

THEOREM 4.1: *The following are equivalent:*

- (a) X is exponential relative to Y .
- (b) Y is exponential relative to X .
- (c) $S(X)$ is exponential.
- (d) $R(Y)$ is exponential.
- (e) $\ast = \circ$.

Moreover, the exponential property defines an equivalence relation on the class of continuous distributions on $[a, b]$.

PROOF: The important fact is that X is exponential relative to Y if and only if X is memoryless on the semigroup $([a, b], \circ)$:

$$R(x \circ y) = R(x) + R(y) \quad \text{for } x, y \in [a, b].$$

Equivalently, $\ast = \circ$ and hence $S = cR$ for some $c > 0$. The other parts of the theorem follow easily. ■

Suppose now that we fix X with cumulative failure rate R and semigroup operator \ast . By Theorem 4.1, the distributions that are exponential relative to X have cumulative failure rate functions of the form cR , $c > 0$. In many cases, it is natural to write the parameter c in the form $R(\lambda)$ where λ is in the support set (a, b) . We will refer to λ as the *exponential parameter* (relative to X). The following theorem now follows easily from Theorem 4.1.

THEOREM 4.2: Suppose that X_1, X_2, \dots, X_n are independent, exponential random variables relative to X with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$. Then

$$\min\{X_1, X_2, \dots, X_n\}$$

is exponential relative to X with parameter $\lambda_1 \ast \lambda_2 \ast \dots \ast \lambda_n$.

Special cases of Theorem 4.2 have appeared in the literature (but not in our terminology, of course).

5. THE IFR PROPERTY

First, suppose that R and S are nonsingular. If the failure rate R'/S' of X relative to Y is increasing, then it is simple to show that the cumulative failure rate R of X is convex on $([a, b], \circ)$:

$$\text{If } x, y, h \in [a, b] \text{ and } x \leq y \text{ then } R(x \circ h) - R(x) \leq R(y \circ h) - R(y). \quad (4)$$

Thus, in general, we will say that X has *increasing failure rate* (IFR) relative to Y if R is convex on $([a, b], \circ)$ and that X has *decreasing failure rate* (DFR) on $([a, b], \circ)$ if R is concave on $([a, b], \circ)$.

THEOREM 5: The following are equivalent:

- (a) X is IFR relative to Y .
- (b) Y is DFR relative to X .
- (c) $S(X)$ is IFR.
- (d) $R(Y)$ is DFR.
- (e) $(x \circ h) \ast y \leq (y \circ h) \ast x$ for $x, y, h \in [a, b]$ and $x \leq y$.

Moreover, the IFR relation defines a partial order on the class of continuous distributions on $[a, b]$, modulo the exponential equivalence.

PROOF: It is straightforward to show that Eq. (4) is equivalent to the convexity of RS^{-1} on $[0, \infty)$. The equivalence of condition (e) then follows from the definition of the semigroup operations. The equivalence of (a)–(c) and the last statement will follow from Theorem 9.2. ■

Intuitively, the conditions in Theorem 5 mean that, if time is transformed so that Y becomes ageless (exponential), then X has increasing failure rate (in the usual sense).

The IFR order is similar (but not equivalent) to two other concepts in the literature (see [2, 6]). Let F and G denote the distribution functions, respectively, of X and Y . First, X is said to be smaller than Y in the *likelihood ratio order* if GF^{-1} is convex on $([0, 1), +)$. Next, X is said to be smaller than Y in the *convex transform order* if $G^{-1}F = S^{-1}R$ is convex on $([a, b), +)$. The likelihood ratio order is a true partial order on the distributions, and so is the convex transform order if $b < \infty$. In particular, if X is smaller than Y and Y is smaller than X (with respect to one of these orders) then $F = G$. On the other hand, the IFR order never defines a partial order on the distributions, but only on the equivalence classes of distributions, modulo the exponential property.

6. THE IFRA PROPERTY

Next, X has *increasing failure rate average* (IFRA) relative to Y if the average failure rate function R/S is increasing on (a, b) . Similarly, X has *decreasing failure rate average* (DFRA) relative to Y if R/S is decreasing on (a, b) .

THEOREM 6: *The following are equivalent:*

- (a) X is IFRA relative to Y .
- (b) Y is DFRA relative to X .
- (c) $S(X)$ is IFRA.
- (d) $R(Y)$ is DFRA.
- (e) $x^{*\alpha} \leq x^{\circ\alpha}$ for $x \in [a, b)$ and $\alpha \geq 1$.

Moreover, the IFRA relation defines a partial order on the class of continuous distributions on $[a, b)$, modulo the exponential equivalence.

PROOF: It is straightforward to show that the definition is equivalent to the condition that $RS^{-1}(t)/t$ is increasing on $(0, \infty)$. The equivalence of condition (e) then follows from the definition of the semigroup operations. The equivalence of (a)–(c) and the last statement follows from Theorem 9.2. ■

Intuitively, the conditions in Theorem 6 mean that, if time is transformed so that Y becomes ageless (exponential), then X has increasing failure rate average (in the usual sense).

The IFRA order is similar (but not equivalent) to two other concepts in the literature (see [2, 6]). First, X is said to be smaller than Y in the *reversed hazard rate*

order if $GF^{-1}(u)/u$ is increasing on $(0, 1)$. Next, X is said to be smaller than Y in the star order if $G^{-1}F(x)/x = S^{-1}R(x)/x$ is increasing on (a, b) .

7. THE NBU PROPERTY

Next, X is *new better than used* (NBU) relative to Y if the conditional reliability function of $x^{o(-1)} \circ X$ given $X \geq x$ is dominated by the reliability function of X :

$$P(X \geq x \circ y | X \geq x) \leq P(X \geq y) \quad \text{for } x, y \in [a, b]. \quad (5)$$

Similarly, X is *new worse than used* relative to Y if the inequality on the probabilities is reversed.

THEOREM 7: *The following are equivalent:*

- (a) X is NBU relative to Y .
- (b) Y is NWU relative to X .
- (c) $S(X)$ is NBU.
- (d) $R(Y)$ is NWU.
- (e) $x * y \leq x \circ y$ for $x, y \in [a, b]$.

Moreover, the NBU relation defines a partial order on the class of continuous distributions on $[a, b]$, modulo the exponential equivalence.

PROOF: It is straightforward to show that the definition in Eq. (5) is equivalent to the condition that RS^{-1} is superadditive on $(0, \infty)$:

$$RS^{-1}(s+t) \geq RS^{-1}(s) + RS^{-1}(t) \quad \text{for } s, t \in (0, \infty).$$

The equivalence of condition (e) then follows from the definition of the semigroup operations. The equivalence of (a)–(c) and the last statement follows from Theorem 9.2. ■

Intuitively, the conditions in Theorem 7 mean that, if time is transformed so that Y becomes ageless (exponential), then X is new better than used (in the usual sense).

The NBU relation is similar (but not equivalent) to another concept in the literature (see [2,6]). Random variable X is said to be smaller than Y in the superadditive order if $G^{-1}F = S^{-1}R$ is superadditive on $[a, b]$.

Of course, the IFR, IFRA, and NBU aging properties are related as in the standard case. That is, X IFR relative to Y implies X IFRA relative to Y which in turn implies X NBU relative to Y . Equivalently, the NBU partial order extends the IFRA partial order which in turn extends the IFR partial order.

8. EXAMPLES

In this section, we will consider several two-parameter families of distributions. In each case, the parameter c determines the exponential parameter in the sense of Section 4, while the parameter α determines the relative aging of the distributions.

The families were chosen because they have a variety of aging properties in the ordinary sense, but in each case the rate of aging of the distribution increases, in the IFR sense of Section 5, as α increases.

Example 8.1: The Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $c > 0$ has reliability function

$$\bar{F}(x) = \exp(-cx^\alpha), \quad x \in [0, \infty).$$

It follows immediately that the corresponding semigroup operation depends only on the shape parameter α and is given by

$$x * y = (x^\alpha + y^\alpha)^{1/\alpha} \quad \text{for } x, y \in [0, \infty).$$

If the parameter c is written as λ^α , $\lambda > 0$, then λ is the exponential parameter. The special case of Theorem 4.2 applied to the Weibull family is frequently mentioned in the literature (see [4], for example).

It is well-known that the Weibull distribution is DFR if $0 < \alpha < 1$, exponential if $\alpha = 1$, and IFR if $\alpha > 1$ (in the ordinary sense). A stronger statement, from Theorem 5, is that a Weibull distribution with shape parameter β is IFR relative to a Weibull distribution with shape parameter α if and only if $\alpha \leq \beta$.

Example 8.2: The modified extreme value distribution with parameters $\alpha > 0$ and $c > 0$ has reliability function

$$\bar{F}(x) = \exp(-c(e^{\alpha x} - 1)), \quad x \in [0, \infty).$$

Again, the corresponding semigroup operation depends only on α and is given by

$$x * y = \frac{1}{\alpha} \ln(e^{\alpha x} + e^{\alpha y} - 1), \quad x, y \in [0, \infty).$$

Thus, two distributions with the same value of α are exponential relative to one another. The distributions are IFR (in the ordinary sense) for any α . On the other hand, from Theorem 5, a distribution with parameter β is IFR relative to a distribution with parameter α if and only if $\alpha \leq \beta$.

Example 8.3: A family of distributions related to the Pareto distribution have reliability function

$$\bar{F}(x) = \left(\frac{\alpha}{x + \alpha} \right)^c, \quad x \in [0, \infty),$$

where $\alpha > 0$ and $c > 0$. Again, the corresponding semigroup operation depends only on α and is given by

$$x * y = x + y + \frac{xy}{\alpha}, \quad x, y \in [0, \infty).$$

Thus, two distributions with the same value of α are exponential relative to one another. The distributions are DFR (in the ordinary sense) for any α . On the other

hand, from Theorem 5, a distribution with parameter β is IFR relative to a distribution with parameter α if and only if $\alpha \leq \beta$.

Example 8.4: Consider the two parameter family of distributions with reliability functions given by

$$\bar{F}(x) = (1 - x^\alpha)^c, \quad x \in [0, 1),$$

where $\alpha > 0$ and $c > 0$. If $c = 1$ or $\alpha = 1$, the distribution is beta; if $c = 1$ and $\alpha = 1$, the distribution is uniform on $(0, 1)$.

Again, the corresponding semigroup operation depends only on α and is given by

$$x * y = (x^\alpha + y^\alpha - x^\alpha y^\alpha)^{1/\alpha} \quad \text{for } x, y \in [0, 1).$$

Thus, two distributions with the same value of α are exponential relative to one another.

In the ordinary sense, these distributions are IFR if $\alpha \geq 1$, but are neither NBU nor NWU if $0 < \alpha < 1$. On the other hand, from Theorem 5, a distribution with parameter β is IFR relative to a distribution with parameter α if and only if $\alpha \leq \beta$.

9. GENERAL AGING PROPERTIES

Consider a general aging property, and the corresponding improvement property, in the standard setting $([0, \infty), +)$. We are interested in characterizing those aging properties that can be extended to relative aging properties for continuous distributions on an arbitrary interval $[a, b)$. Moreover, we want the relative aging property to define a partial order on the distributions, modulo the exponential equivalence, just as the IFR, IFRA, and NBU properties do. In light of Theorems 5, 6, and 7, such a characterization would seem naturally to describe "strong" aging properties. Suppose that the following axioms hold for our aging property on $([0, \infty), +)$:

AXIOMS 9.1:

- (a) *A distribution both ages and improves if and only if the distribution is exponential.*
- (b) *The distribution with cumulative failure rate R ages if and only if the distribution with cumulative failure rate R^{-1} improves.*
- (c) *If the distributions with cumulative failure rates R and S age, then the distribution with cumulative failure rate RS ages.*
- (d) *If a distribution is IFR then the distribution ages.*

The last axiom is to ensure that our general notion really does capture some aspect of aging and to incorporate the idea that the IFR property is the strongest aging property.

Suppose now that X and Y are continuous random variables supported on the closure of $[a, b)$ and let R and S , respectively, denote the cumulative failure rate

functions of X and Y , as in Eq. (1). We will say that X ages relative to Y if the distribution on $[0, \infty)$ with cumulative failure rate RS^{-1} ages. Similarly, we will say that X improves relative to Y if the distribution with cumulative failure rate RS^{-1} improves.

THEOREM 9.2: *Suppose that Axioms 9.1 hold. The following are equivalent:*

- (a) X ages relative to Y .
- (b) Y improves relative to X .
- (c) $S(X)$ ages.
- (d) $R(Y)$ improves.

Moreover, the aging relation defines a partial order on the class of continuous distributions on $[a, b)$, modulo the exponential equivalence.

PROOF: The equivalence of (a)–(c) follows from the definitions and 9.1(b). The partial order property, modulo the exponential equivalence follows from 9.1(a)–(c).

THEOREM 9.3. *The IFR, IFRA, and NBU aging properties satisfy Axioms 9.1.*

Because of Theorems 9.2 and 9.3, Axioms 9.1 seem very natural for a strong axiom property. Axioms (a) and (d) are commonly used in the literature to justify new aging properties, but Axioms (b) and (c) seem to have been overlooked, even though they are critical for partial order property. This fact, for us, is strong evidence of the value of this point of view.

Not all of the common aging properties on $([0, \infty), +)$ are strong in the sense of Axioms 9.1. In particular, the New Better than Used in Expectation (NBUE) property

$$E(X - t | X > t) \leq E(X) \quad \text{for } t \geq 0 \quad (7)$$

fails to satisfy 9.1(b) (although it is well known that it satisfies 9.1(a)).

Example 9.4: Let $a > 0$, $b > 0$, $c > 0$ and let R be the cumulative failure rate function given by

$$R(t) = \begin{cases} at, & 0 \leq t < 1 \\ a + b(t - 1), & 1 \leq t < 2 \\ a + b(t - 1) + c(t - 2), & t \geq 2 \end{cases}$$

It is straightforward to show the following: If

$$b < a < c, \quad (8)$$

then the distribution associated with R is not NBU and, hence, not IFR or IFRA. If in addition,

$$e^{-b} \geq \frac{c(b-1)}{a(c-b)}, \quad (9)$$

then the distribution is NBUE. Finally, if

$$\frac{a-b}{c-b} > e^{-1}, \quad (10)$$

then the distribution with cumulative failure rate R^{-1} is not NWUE. If $a = 0.66$, $b = 0.22$, $c = 1.32$ then conditions (8), (9), and (10) are satisfied and thus, for these parameter values, the distribution with cumulative failure rate R is NBUE, but the distribution with cumulative failure rate R^{-1} is not NWUE. Thus, the NBUE aging property is not strong in the sense of Axioms 9.1.

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